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Universal configurational structure in two-dimensional scalar models

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Abstract. Monte Carlo methods are used to explore the universal configurational structure of two-dimensional spin-$\frac{1}{2}$, spin-1 and border-ϕ^4 models. Comparison of spin-$\frac{1}{2}$ and spin-1 data provides evidence that the magnetisation distribution (effectively the Helmholtz free-energy function) and its coupling derivative (effectively the internal-energy function) constitute readily accessible signatures of a universality class. It is shown that, when allowance is made for relatively large corrections-to-scaling effects, the behaviour of the border-ϕ^4 model may be satisfactorily matched to that of the other two models, substantiating the view that the border model does indeed belong to the Ising universality class.

1. Introduction

In the vicinity of a continuous phase transition, certain physical observables display a remarkable degree of insensitivity to the microscopic details of the physical system involved. This is the phenomenon of universality. The phenomenon is expressed in its fullest form in the claim (Bervillier 1976) that all (multi-point) correlation functions are universal, provided the variables they describe are separated by large enough distances, and provided the scales of the variables and their separation are appropriately normalised. The full extent of this claim is seldom put to explicit test. It is customary, rather, to focus on a few prototype quantities, whose universality is implied by the general claim, most notably the critical indices, and, less frequently, appropriate ratios of the amplitudes of critical singularities.

The universality claim may, however, be put to a considerably more exacting test. The universal scaling properties of the multi-point correlation functions imply corresponding properties for the distribution $P(M_l)$ of the coarse-grained ordering variable, with coarse-graining length l (Bruce 1981). This distribution is directly accessible to Monte Carlo techniques. It has been the subject of a number of studies. Binder (1981) investigated how the form of the distribution depends upon the nature of the blocking used to implement the coarse graining: the form differs according to whether the blocks are small subunits of a much larger system ($l \ll L$, the linear dimension of the system) or actually constititute the system itself ($l = L$), in which case the form of the distribution turns out to depend upon the nature of the boundary conditions employed. Binder also showed how the scaling properties of the moments of the distribution may be used to compute critical indices, determine the surface tension (Binder 1982) and locate phase boundaries (Binder and Landau 1984). Complementary analytic work on the moments of the distribution (for $l = L$) has also been reported for scalar models, using the ε expansion in three dimensions (Brézin and Zinn-Justin 1985) and conformal invariance in two dimensions (Burkhardt and Derrida 1985). The former study has recently been extended (Eisenriegler and Tomaschitz 1987) to yield a result for the $l = L$ distribution itself, within a first-order ε expansion. The idea that the distribution might be usefully employed as a hallmark of a universality class has also been developed briefly (Bruce 1985, hereafter referred to as I) in an attempt to resolve the controversial question (Baker and Johnson 1984) of whether or not the two-dimensional ϕ^4 model, at its 'border' critical point, falls into the Ising universality class.

In this paper we extend this programme with further Monte Carlo studies of the order-parameter distribution $P(M_L)$ in two-dimensional systems, of linear dimension L, with a scalar order parameter. Firstly, we check the proposed universality of the distribution, at the bulk system critical point, through explicit studies of spin-$\frac{1}{2}$ and spin-1 Ising models, whose membership of a common ('Ising') universality class seems undisputed. The associated fixed-point function, $P^*(M_L)$, thus identified, may alternatively be viewed as a measure of the Helmholtz function $A(M_L)$ for the Ising universality class, in the (finite-size) critical limit. Secondly, we determine the form, and check the universality, of the function which controls the deviation of $P(M_L)$ from its critical fixed-point limit $P^*(M_L)$, when the system is not precisely at its bulk critical point. This function turns out to be effectively a measure of the internal-energy function $E(M_L)$ for the Ising universality class, in the (finite-size) critical limit. Thirdly, we determine the form of the ('corrections-to-scaling') function which controls the deviation of $P(M_L)$ from $P^*(M_L)$ when the system is at the bulk critical point, but the system size L is not sufficiently large to provide access to the asymptotic regime in which pure fixed-point behaviour is displayed. Finally, armed with these three functions, each of which may, we believe, be regarded as characteristic of the Ising universality class, we consider the ϕ^4 'border' model. We find that the bulk of the discrepancy between the border-model critical-point distribution, computed in I, and the Ising fixed-point distribution should be attributed to corrections to scaling.

2. Background

The models to be considered in this paper are characterised by a set of local variables ϕ_j, $j = 1, \ldots, L^d$, each variable being associated with one of the L^d sites of a (square $d = 2$) lattice of linear dimension L and subject to periodic boundary conditions. The statistical properties of the models are prescribed by the corresponding configurational probability which may be written in the form

$$P(\{\phi\}) = Z^{-1} \prod_j (p_0(\phi_j)) \exp[-KE(\{\phi\})] \tag{2.1a}$$

where

$$E(\{\phi\}) = -\sum_{\langle ik \rangle} \phi_i \phi_k \tag{2.1b}$$

is the nearest-neighbour interaction energy (in units of the bond strength) and

$$Z = \prod_j \left(\int \mathrm{d}\phi_j \, p_0(\phi_j) \right) \exp[-KE(\{\phi\})] \tag{2.1c}$$

is the partition function. The models are distinguished by the form assigned to the local spin weight function

$$p_0(\phi_j)=\begin{cases}\delta(\phi_j-1)+\delta(\phi_j+1) & (2.2a)\\ \delta(\phi_j-1)+\delta(\phi_j+1)+\delta(\phi_j) & (2.2b)\\ \exp(-A\phi_j^2-B\phi_j^4) & (2.2c)\end{cases}$$

corresponding, respectively, to the spin-$\frac{1}{2}$, spin-1 and ϕ^4 models. The additional constraint $A=0$ and the convention $B=0.114\,233\ldots$ (normalising p_0 to unit variance), together serve to locate the so-called border model in ϕ^4 parameter space (Baker and Johnson 1984).

In this paper we shall be concerned with the equilibrium behaviour of the magnetisation variable defined by

$$M_L=L^{-d}\sum_j \phi_j. \tag{2.3}$$

We shall focus on the equilibrium probability density

$$P(M_L)\equiv\left\langle\delta\left(M_L-L^{-d}\sum_j\phi_j\right)\right\rangle \tag{2.4}$$

which may be regarded as a measure of the Helmholtz potential

$$F(M_L)=-\ln Z(M_L)$$

where $Z(M_L)$ is the partition function in the constant magnetisation ensemble:

$$Z(M_L)=\prod_j\left(\int \mathrm{d}\phi_j\,p_0(\phi_j)\right)\exp[-KE(\{\phi\})]\delta\left(M_L-L^{-d}\sum_j\phi_j\right). \tag{2.5}$$

Specifically,

$$P(M_L)=Z^{-1}\exp[-F(M_L)] \tag{2.6a}$$

or

$$F(M_L)-F(0)=-\ln(P(M_L)/P(0)). \tag{2.6b}$$

We shall also find it useful to examine the coupling derivative (effectively the temperature derivative) of the magnetisation distribution,

$$P_1(M_L)\equiv\partial P(M_L)/\partial K \tag{2.7}$$

which is effectively a measure of the internal-energy function

$$E(M_L)=\frac{\partial F(M_L)}{\partial K}\equiv-\frac{\partial\ln Z(M_L)}{\partial K}. \tag{2.8}$$

Specifically, recalling (2.6a),

$$P_1(M_L)=P(M_L)[E-E(M_L)] \tag{2.9a}$$

where

$$E\equiv-\partial\ln Z/\partial K\equiv\langle E\{\phi\}\rangle \tag{2.9b}$$

is the mean energy in units of the coupling strength.

We will focus on the behaviour of these functions in the vicinity of a critical point, located by a critical coupling K_c. The universal scaling properties of the multi-point correlation functions (Bervillier 1976) imply for the distribution $P(M_L)$ the limiting (large L, small $K-K_c$) form (Binder 1981, Bruce 1981)

$$P(M_L) \approx b(L)\tilde{P}(b(L)M_L, a(L)(K-K_c), e(L)) \tag{2.10}$$

where

$$b(L) = b_0 L^{\beta/\nu} \tag{2.11a}$$

$$a(L) = a_0 L^{1/\nu} \tag{2.11b}$$

$$e(L) = e_0 L_0^{-\omega} \tag{2.11c}$$

while a_0, b_0 and e_0 are non-universal constants. The form assigned to the scale factor $b(L)$ guarantees that the variance of the critical-point distribution has the asymptotic (large-L) behaviour $\langle M_L^2 \rangle \sim L^{-2\beta/\nu}$, which is mandated by the finite-size-limited behaviour of the two-point correlation function. The form assigned to the scale factor $a(L)$ ensures that the right-hand side of (2.10a) depends upon the bulk system correlation length $\xi \sim |K-K_c|^{-\nu}$ through the appropriate scaling combination L/ξ. Finally, the factor $e(L)$ incorporates the effects of the leading (least-irrelevant) correction-to-scaling field.

To the extent that L is large enough on microscopic scales (and that $e(L)$ is thus 'small') and ξ is large compared to L (so that $a(L)$ $(K-K_c)$ is 'small'), we may expect that the right-hand side of (2.10) can be expanded to give

$$P(M_L) \approx b(L)[P^*(\tilde{M}_L) + a(L)(K-K_c)P_1^*(\tilde{M}_L) + e(L)P_2^*(\tilde{M}_L) + \ldots] \tag{2.12a}$$

where we introduce the scaled magnetisation variable

$$\tilde{M}_L = b(L)M_L. \tag{2.12b}$$

Given appropriate choices of the parameters a_0 and b_0, the function $\tilde{P}$ appearing in equation (2.10) should have a form unique to a universality class, in the large-L limit, where the correction-to-scaling field $e(L)$ is negligible. In particular the functions P^* and P_1^* appearing in (2.12a) should also have universal forms. A stronger claim may sometimes be warranted: to the extent that the members of a universality class are characterised by a unique correction-to-scaling index ω (and thus, implicitly, a unique leading correction-to-scaling field), the function P_2^* in (2.12a) should also be unique to a universality class.

3. Monte Carlo studies

3.1. *Computational details*

The Monte Carlo studies reported here were performed using a Metropolis algorithm implemented in parallel on an ICL distributed array processor. General features of parallel coding strategy are discussed by Bowler and Pawley (1984). The systems studied were of linear dimensions ranging from $L = 8$ to $L = 64$, with periodic boundary conditions. The basic observables selected were the probability distribution $P(M_L)$ (which determines the Helmholtz function $F(M_L)$: equation (2.6b)) and the energy function $E(M_L)$ (which determines the coupling derivative, $P_1(M_L)$, of the distribution $P(M_L)$: equation (2.9a)). The distribution $P(M_L)$ was determined (initially as a histogram) directly in accordance with its definition (2.4). The energy function $E(M_L)$ was determined as an average, for each M_L value explored in the course of the simulation, of the associated values of the interaction energy (2.1*b*). The two functions were measured for each model type, system size and coupling of interest. The spin-$\frac{1}{2}$ and spin-1 model studies which constitute the bulk of the new work reported here consisted of 2×10^5 lattice passes for equilibrium, followed by a sequence of 2×10^5 observations with between $2(L=8)$ and $80(L=64)$ lattice passes between each observation. In each instance the whole procedure was repeated a number (up to 16) of times, the statistical independence of the data sets thus generated was tested and statistical error limits assigned on the basis of the appropriate variance. The studies of the ϕ^4 model presented here represent extensions of work reported in I. The form of Metropolis algorithm deployed in dealing with continuously distributed variables is discussed in that paper. The results presented here were accumulated over some 2×10^4 observations separated by 10^2 lattice passes, after in excess of 10^6 passes for equilibrium in the case of the largest $(L=64)$ lattice.

3.2. *The moments of the distribution*

To determine the fixed-point distributions P^* and P_1^*, it is clearly essential to be able to assign a precise value to the critical coupling. The value of the critical coupling for the spin-$\frac{1}{2}$ Ising model is known from Onsager's solution: $K_c^{(1/2)}=0.4406\ldots.$ For the ϕ^4 border model, Bruce (1985) assigned the value $K_c^{\mathrm{BM}}=0.3282(2)$, while Baker and Johnson (1984) found $K_c^{\mathrm{BM}}=0.330$. For the spin-1 Ising model the most recent series estimates (Adler and Enting 1984) give $K_c^{(1)}=0.590\,48$ $(^{-25}_{+47})$. We proceed to describe a check on this assignment, based upon an analysis of the moments $M_L^{(n)}\equiv\langle M_L^n\rangle$ of the distribution function $P_L(M)$. We follow (and thus implicitly check) the strategy of I. Specifically, we focus on the combinations

$$G_L\equiv[3(M_L^{(2)})^2-M_L^{(4)}]/2(M_L^{(2)})^2 \tag{3.1a}$$

and

$$R_L\equiv M_L^{(4)}/M_L^{(2)}. \tag{3.1b}$$

Table 1 presents the values of these quantities obtained from the distribution $P(M_L)$ (measured as described in the preceding section) for the spin-1 model at a range of

Table 1. Characteristic parameters (equations (3.1*a*, *b*)) of the spin-1 model for a variety of system sizes L and couplings strengths K.

L	K	G_L	R_L
8	0.590 48	0.910 54 (19)	0.648 14 (47)
16	0.588 5	0.899 56 (31)	0.519 02 (51)
16	0.590 48	0.913 56 (34)	0.535 26 (55)
16	0.592 5	0.923 26 (31)	0.546 91 (65)
32	0.589 5	0.899 31 (64)	0.432 22 (124)
32	0.590 48	0.916 35 (66)	0.448 14 (122)
32	0.591 5	0.922 52 (55)	0.455 91 (107)
64	0.590 0	0.909 27 (180)	0.370 19 (64)
64	0.590 48	0.916 74 (146)	0.377 22 (12)
64	0.591 0	0.923 44 (135)	0.384 61 (72)

couplings including the series estimate for the critical coupling. These results were analysed within the framework of the expansion (2.12a) which implies the forms

$$G_L = G^*[1 + g_1 L^{1/\nu}(K - K_c) + g_2 L^{-\omega} + \ldots] \tag{3.2a}$$

$$R_L = r_0 L^{-2\beta/\nu}[1 + r_1 L^{1/\nu}(K - K_c) + r_2 L^{-\omega} + \ldots] \tag{3.2b}$$

given the assignments (2.11a–c) and (2.12b).

Implementing these equations directly in the analysis of table 1 we found values of the primary critical indices consistent with the expected Ising universality class results. The critical coupling was found to be $K_c^{(1)} = 0.590\,35\,(30)$, consistent with the series assignment noted above. The best fit value of the correction-to-scaling index was found to be $\omega = 1.4$, consistent with the conjecture ($\omega = \frac{4}{3}$) of Nienhuis (1982) but with a large uncertainty. The value assigned to the fixed-point coupling constant G^* was found to be strongly correlated with the value assigned to ω. Fixing $\omega = \frac{4}{3}$ we found $G^* = 0.9149\,(44)$, in accord with the assignment $G^* = 0.916\,(1)$ made in an earlier analysis of the spin-$\frac{1}{2}$ Ising system (Bruce 1985). To tighten these assignments we explored one further strategy. We incorporated into the data set for the fitting analysis the estimate $G^* = 0.9154\,(10)$ obtained from further studies of the spin-$\frac{1}{2}$ Ising system for $L = 64$ (at the exact critical coupling) on the assumption (warranted by our earlier studies of smaller-L systems) that corrections to scaling are small and the approach to the fixed point is rapid in this instance. With this additional input we found $K_c^{(1)} = 0.590\,33\,(7)$, $G^* = 0.9145\,(6)$ and $\omega = 1.33\,(+0.55, -0.27)$, with a χ^2 value of 6.4 for six degrees of freedom.

In summary, the analysis is fully consistent with the existence of a universal value for the fixed-point cumulant ratio, with the conjectured value of the correction-to-scaling index ω, and with the series estimate for the critical coupling.

3.3. *The critical limit: the function P^**

Given the approximation already made in the course of the analysis of the moments, the fixed-point distribution P_1^* for the 2D Ising universality class should be well approximated by the magnetisation distribution for the $L = 64$ spin-$\frac{1}{2}$ model. The form of this distribution is shown in figure 1(a). We have chosen the (non-universal) scale factor b_0 (equations (2.11a) and (2.12b)) so that the distribution has unit variance. The figure also shows the result obtained for the $L = 64$ spin-1 model at the value for the critical coupling ($K_c^{(1)} = 0.590\,48$) suggested by series expansions. The agreement is strikingly good. Indeed, it is better than one might expect given that we have taken account neither of corrections to asymptotic scaling behaviour, nor of the $K_c^{(1)}$ refinement suggested by our moments analysis. We will consider the corrections associated with these effects in due course.

In figure 1(b) we make a similar comparison between the form assigned to P^* on the basis of the spin-$\frac{1}{2}$ Ising calculation and that obtained directly from the form of the $L = 64$ magnetisation distribution of the border model with the assignment (made in I) $K_c^{BM} = 0.328\,26$. The agreement is substantial but not as impressive as that in figure 1(a). Given our central thesis that the distribution P^* is a signature of a universality class, it is clear that, if the claim that the border model does fall into the Ising universality class is to be warranted, the discrepancies must be traced either to the effects of corrections to scaling or to the need for further refinement of the critical coupling. We shall pursue this point in § 3.6.

We conclude this section with two further observations.

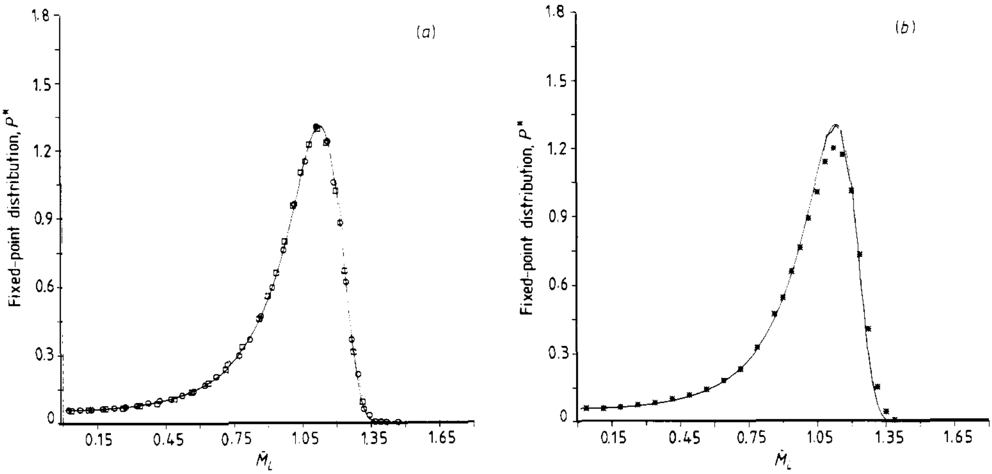

Figure 1. (a) Estimates of the fixed-point order-parameter distribution P^* for the 2D Ising universality class based on the $L=64$ spin-$\frac{1}{2}$ magnetisation distribution at the exact critical point $K_c^{(1/2)}$ (○) and the $L=64$ spin-1 magnetisation distribution at the series critical-point estimate $K_c^{(1)} \doteq 0.590\,48$ (□). The statistical uncertainties do not exceed the symbol sizes. The full curve provides a smooth representation of the spin-$\frac{1}{2}$ data. (b) Estimates of the fixed-point distribution P^* in the $L=64$ border-ϕ^4 magnetisation distribution at $K_c^{\mathrm{BM}} = 0.328\,26$ (*) together with the spin-$\frac{1}{2}$ estimate (——). These results make no allowance for corrections to scaling.

Firstly, we note the existence of an exact result for the large-M_L limit of the probability distribution function $P(M_L)$ for the spin-$\frac{1}{2}$ Ising model at its bulk critical point (McCoy and Wu 1973) from which one may infer that, for large $\tilde{M}_L$,

$$P^*(\tilde{M}_L) \sim \exp(-p_0 \tilde{M}_L^{\delta-1}) \tag{3.3}$$

where $\delta = 15$ is the index characterising the variation of the magnetisation with a magnetic field along the critical isotherm. Our results are consistent with this form: fitting the observed P^* with the function (3.3) in the range $\tilde{M} > 1.18$ gives $\delta = 15.22$ (18) with a χ^2 of 0.74 for three degrees of freedom.

Secondly, we note that the key result of this section, the structure of the fixed-point distribution P^*, may be usefully recast as a statement of the form of the Helmholtz free-energy function $F(M_L)$ at criticality. The result for $\Delta F(M_L) \equiv F(M_L) - F(0)$ (implied directly by equations (2.6b) and (2.12a) and figure 1(a)) is shown in figure 2(a).

3.4. *Deviations from criticality: the function P_1^**

The function P_1^* controls the effects of deviation from criticality upon the magnetisation probability distribution. As such it may be computed from the results of simulations at small but finite $K - K_c$. Alternatively it may be computed from simulations *at* criticality by appeal to the relation (2.9a). We have implemented both approaches. The results are fully consistent with one another, thereby vindicating the analyticity assumption made in inferring (2.12a) from (2.10). We present only the results derived from the latter strategy, which is statistically superior.

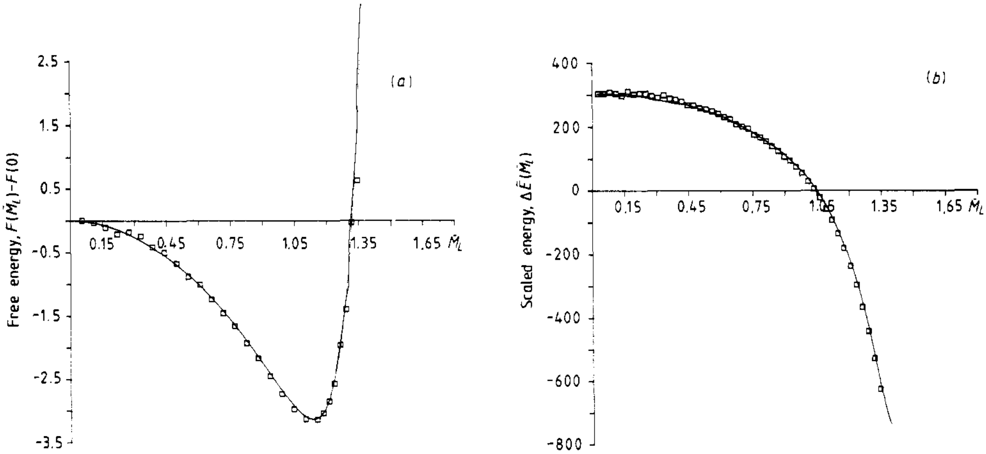

Figure 2. (*a*) The critical-point free-energy function difference defined by equation (2.6*b*) for spin-$\frac{1}{2}$ (——) and spin-1 (□) systems, computed from the results shown in figure 1(*a*). (*b*) The critical-point scaled-energy function difference defined by equation (3.4) for spin-$\frac{1}{2}$ (——) and spin-1 (□) systems.

The results may, in the first instance, be expressed in terms of the scaled energy function,

$$\Delta\tilde{E}(\tilde{M}_L) = a^{-1}(L)(E(M_L) - E)_{K=K_c} \tag{3.4}$$

where $E(M_L)$ (the mean energy of configurations with a given M_L) and E (the mean energy over all configurations) are defined in (2.8) and (2.9*b*). The amplitude factor $a(L)$, inserted for convenience in (3.4), is that introduced in (2.10). Inspection of equations (2.7), (2.9*a*) and (2.12*a*) shows that this scaled energy function should be universal modulo the convention used to prescribe the system-specific amplitude a_0 (equation (2.11*b*)), given the continued operation of the 'unit variance' convention prescribing b_0.

Figure 2(*b*) shows the form of the energy function (3.4) computed for studies of the spin-$\frac{1}{2}$ model at criticality (full curve). The amplitude a_0 has been chosen so that $a(L) = 1$ for the $L = 64$ lattice. The figure also shows (data points) the corresponding results for the spin-1 model. The two sets of results have been brought into coincidence by a single scaling (assignment of a_0) confirming the anticipated universality.

The form of the function P_1^* follows immediately from (2.12*a*), (2.7), (2.9*a*) and (3.4). Figure 3(*a*) shows the results obtained for the spin-$\frac{1}{2}$ and spin-1 models. Note that, given the convention we have chosen to prescribe the parameter a_0, the function shown gives an unscaled representation of the coupling derivative P_1 (equation (2.7)) for the $L = 64$ spin-$\frac{1}{2}$ model. Figure 3(*b*) makes a similar comparison between the results obtained from spin-$\frac{1}{2}$ and border-ϕ^4 models. The level of agreement is very similar to that apparent in figure 1(*b*), although the statistical quality of the P_1^* data is considerably poorer.

3.5. *Corrections to scaling: the function P_2^**

The function P_2^* controls the rate at which the scale-invariant fixed-point distribution of the magnetisation is approached with increasing system size, at criticality. Although

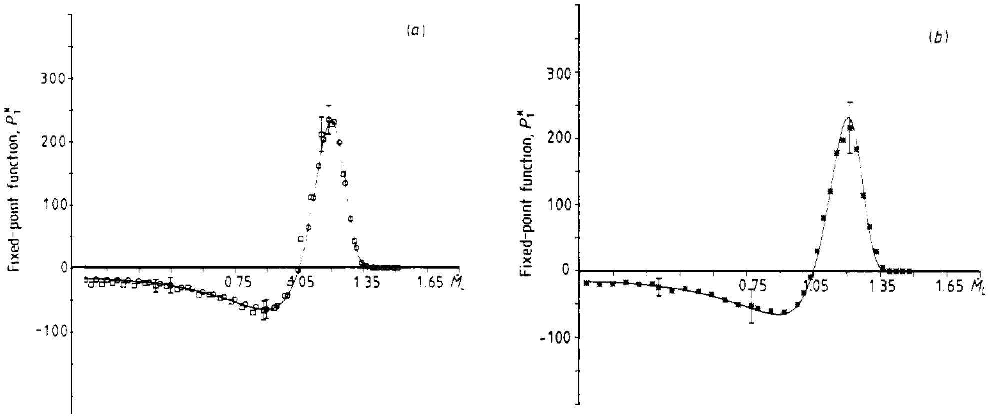

Figure 3. The coupling derivative function P_1^* (equation (2.12*a*)) obtained by computation of the energy operator $E(M)$ (equation (2.9*a*)) for spin-$\frac{1}{2}$ (○) and spin-1 (□) systems. Representative (one standard deviation) statistical uncertainties are indicated. The full curve provides a smooth representation of the spin-$\frac{1}{2}$ data. The units are such that the spin-$\frac{1}{2}$ data give an unscaled representation of the coupling derivative P_1 (equation (2.7)) for the $L=64$ spin-$\frac{1}{2}$ system. (*b*) The coupling derivative function P_1^* for the border-ϕ^4 model (*), compared with the spin-$\frac{1}{2}$ estimate (——).

it is possible to conceive of more sophisticated ways of determining this function (analogous to that by which we determined the function P_1^*) we report here the results obtained by the direct expedient of comparing the forms of distributions for different system sizes. We have focused on the spin-1 and border-ϕ^4 models: the corrections to scaling in the spin-$\frac{1}{2}$ model are 'small' (the approach to the asymptotic limit fast) and, indeed, may not be prototypical of the class of scalar models (Barma and Fisher 1984). To eliminate the uncertainties associated with the two models we have (in each case) determined the corrections-to-scaling function by combining the results for the distributions on three system sizes, at the same near-critical coupling, in such a way as to eliminate the first two terms appearing in the expansion (2.12*a*) (given the assumption $\nu=1$ in (2.11*b*)). The results (given the assumption $\omega=\frac{4}{3}$ in (2.11*c*)) are shown in figure 4. The units have been chosen so that the data shown give the contributions made by corrections to scaling to the respective $L=64$ magnetisation distributions.

As is apparent from the representative error bars, the quality of the data is poor, reflecting the relative enhancement of uncertainties arising from the differencing procedure. Nevertheless, the broad agreement between the two data sets is apparent. Inspection of the figure also reveals the particular importance of corrections to scaling in the border model: the corrections are some 2.5 times larger for the border model than they are for the spin-1 model.

3.6. *The critical limit revisited*

We now return to consider the structure of the fixed-point magnetisation distribution. As noted in §3.3, the data presented in figures 1(*a*) and 1(*b*) represent only approximations to the fixed-point distribution, deviating from it by virtue of corrections to scaling and (possibly) corrections due to residual error in the assignment of the critical coupling. Utilising our results for the coupling derivatives of the distributions (§ 3.4) and the

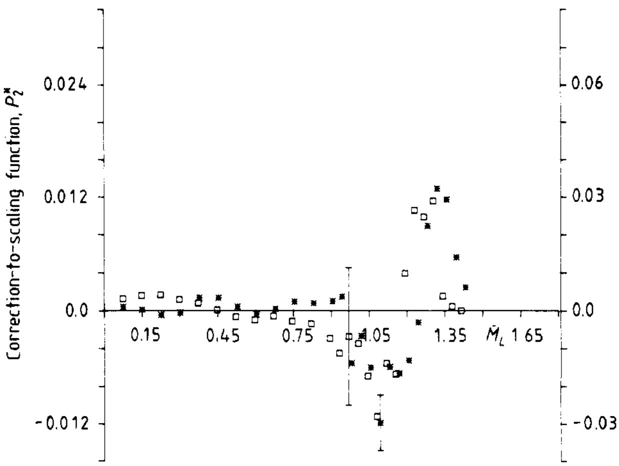

Figure 4. Estimates of the corrections-to-scaling function P_2^* (equation (2.12a)) obtained by comparison of distributions computed on different system sizes for the spin-1 model (□; left-hand scale) and the border model (*; right-hand scale). The units are such that the functions shown give estimates of the contribution which corrections-to-scaling make to the respective $L = 64$ magnetisation distributions.

corrections to scaling (§ 3.5) we can now refine our results somewhat, for both border-ϕ^4 and spin-1 systems. The results of this refinement are shown in figure 5. In each case we have refined the data of figure 1 in two respects. Firstly, we have eliminated the correction-to-scaling contributions, previously identified. Secondly, we have allowed ourselves the license of adjusting our estimates of the critical couplings within the limits prescribed by the results of our moments analysis (§ 3.2 and I). The results

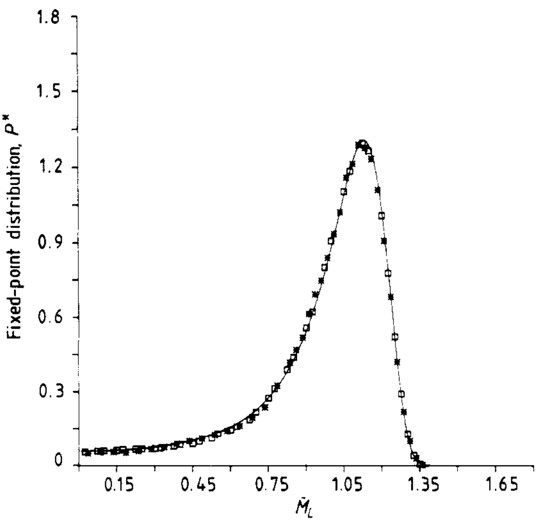

Figure 5. Estimates of the fixed-point distribution P^* for spin-1 (□) and border-ϕ^4 model (*) with allowance made for corrections to scaling, and with the choices $K_c^{(1)} = 0.590\,43$ and $K_c^{BM} = 0.328\,36$. The full curve represents the spin-$\frac{1}{2}$ estimate (figure 1(a)) with no allowance made for corrections to scaling.

shown in figure 5 corresponding to critical coupling values $K_c^{(1)} = 0.590\,43$ and $K_c^{BM} = 0.328\,36$. Neither coupling-constant refinement is statistically significant, resulting in changes in our estimates of the limiting distributions that are smaller than the statistical uncertainties associated with the basic distributions. Nevertheless, one observes that (this is of course the motivation), with these choices, our estimates of the fixed-point distribution based on spin-1 and border-model data are brought into close coincidence with one another, and with the estimate obtained from the spin-$\frac{1}{2}$ model.

4. Summary

The results presented in this paper provide clear evidence in support of the general contention that the order-parameter distribution in a finite near-critical system offers a useful hallmark of a universality class. We have seen explicitly that this contention is valid both *at* the critical point (the universality of the function P^*) and *near* the critical point (the universality of the coupling derivative function P_1^*). There is little doubt that it will hold more generally provided only that L and ξ are both large compared to all other lengths.

Our results also lend support to the view (Barma and Fisher 1984, 1985) that the border-ϕ^4 model belongs to the Ising universality class, and that indications to the contrary (Baker and Johnson 1984) are to be attributed to substantial corrections to scaling. The analysis of our spin-1 data provides some evidence of a correction-to-scaling index ω consistent with the conjecture of Nienhuis (1982). It may prove possible to test this claim more definitely if one can determine the scaling (L dependence) of the irrelevant scaling field by direct simulation measurement of an appropriate operator, instead of appealing, as we have done here, to the intrinsically noisy difference between simulations on different system sizes.

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Note added in proof. In a recent paper, Adler (1987 *J. Phys. A: Math. Gen.* **20** 3419) has presented the results of further series expansion studies of the border model, suggesting, in accord with the conclusions reached here, a critical coupling significantly lower than the earlier series estimate.

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